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Bell's inequality and a strict assessment of the concept of "possession" *

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Abstract

It is argued that the concept of "physical quantities possessed by the system" is both redundant and inappropriate. We have examined two versions of the concept of "possessed values": one identical with the observed values and the other non-identical. Assuming the existence of such "physical quantities possessed by the system" and subjecting it to the framework of Bell theorem under very generous condition allowing even certain form of "nonlocality", one can still arrive at the proper Bell's inequality. With the experimental falsification of Bell's inequality, we conclude that the concept of "possessed values" finds no place in proper physical reasoning.

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1 Introduction

Discussions on the interpretation of quantum mechanics paved a way for a new discipline called meta-dynamics which deals with the foundation of the physical theories. Experimental metaphysics, as baptized by A. Shimony, shows the fundamental aspects of the universal descriptional system called dynamics. Especially Bell theorem[1] verifies that the fundamental issues themselves can be analyzed as those in usual physical problems, especially in prediction-experiment manner. In that sense, Bell theorem conforms a summit of meta-dynamical investigations. However, its interpretation is not unanimous in that its premises are not examined sufficiently enough to everybody's satisfaction.

It has widely been known that the concept of "possession by the system" is problematique in quantum mechanics. But the conception of "reality", which is apt to be identified with the existence of values of physical quantities within the system, is so strongly engraved in human imagination that we could not easily discard the concept of "possession of physical values by the system". The concept of nonlocality might be a convenient, if vague, concept to compromise to save the inconsistency between the quantum mechanics and this conception of reality.

Bell theorem gives a way of experimental approach to some important philosophical issues in physics, such as the concept of "locality" and "reality". The experimental falsification[3] of Bell's inequalities is regarded as a verification of the failure of the theory based *both* on "locality" and on "reality". Though the latter is not the fully examined conception, especially for quantum mechanics, the corroboration of quantum-mechanical prediction in experiment is usually interpreted as a vindication of the existence of "nonlocality" in Nature.

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However, both the notion of locality and reality must be defined more precisely, since the proof, and accordingly the interpretation, of Bell theorem might depend on them.

On the basis of such analysis, we argue in this paper that the concept of "physical quantities possessed by the system" is both redundant and inappropriate. It is redundant because it has no place in the workings of dynamical theories other than its vague implication as the "state of the system", which is well defined on its own stance, and it is inappropriate because it brings contradiction in the light of Bell's inequality. Two versions of the concept of "possessed values" are examined: one identical with the observed values and the other non-identical. Assuming the existence of such "physical quantities possessed by the system" and subjecting it to the framework of Bell theorem under very generous condition allowing even certain form of "nonlocality", one can still arrive at the proper Bell's inequality. More specifically, the concept of nonlocality here is divided into two categories, the ontological and the epistemological one. The former is shown to be perfectly permitted in the derivation, but the latter is not in one version of the "possessed values". But since the epistemological non-locality defined here is highly unacceptable, and the Bell's inequality is shown not to be respected universally, it is safe to conclude that the concept of "possessed values" is in conflict with the proper physical reasoning.

In next section we give a short review of Bell theorem, with a view to fixing the situation and summarizing the received view on it. It is followed by a meticulous examination of the consequence of assuming that the system "possess" the values of the physical quantities. (The appendix shows the detailed steps of the proofs.) The final section discusses the meaning of reality of physical quantities as well as elaboration of the concept of nonlocality in two categories, viz., ontological and epistemological ones. It leads to an alternative interpretation of Bell theorem.

2 Received view on Bell theorem

Consider a composite system made of two two-state subsystems, such as two spin 1/2 particles or two polarized photons. Let $\mathbf{A}$, $\mathbf{B}$ be two physical quantities restricted to each subsystem and α, β be their adjustable parameters representing the measurement setting, such as the direction of the inhomogeneous magnetic field in Stern-Gerlach apparatus or the angle of the analyzer for two-photon cascade experiment. Denote by A_α the presumed values of observables $\mathbf{A}$ with the parameters of measuring apparatus being α and by a_α the corresponding outcome of the measurement. For simplicity we abbreviate A_α, B_β, $A_{\alpha'}$, $B_{\beta'}$ by A, B, A', B' and the corresponding outcomes by a, b, a', b'. We here are taking, deliberately, a most generous position that the measured values a, b, a', b' may not be identical with the actually possessed values A, B, A', B'. (Throughout this paper, $\mathbf{A}$ etc. denotes the physical quantity, A etc. the possessed value of $\mathbf{A}$, a etc. the measured value.)

Bell theorem concerns with the coincidence rate defined by

$$\xi(a, b) = \frac{\sum abN(a, b)}{N_{\rm tot}} \ , \tag{1}$$

where $N(a, b)$ means the number of the joint event that the outcomes of two measuring apparatus are given by a and b, respectively. $N_{\rm tot} = \sum N(a, b)$ is the total number of trials which is usually, but not necessarily, taken as infinity. Other coincidence rates $\xi(a, b')$, $\xi(a', b)$, and $\xi(a', b')$ are defined similarly.

[Bell theorem]

If we assume the locality and the reality of physical quantities, the particular combination of the

coincidence rates defined by

$$\mathcal{B} := |\xi(a,b) - \xi(a,b')| + |\xi(a',b) + \xi(a',b')|$$

satisfies the following (Bell's) inequality

$$\mathcal{B} \leq 2 \; .$$

The inequality above, called CHSH(Clauser-Horne-Shimony-Holt) inequality[4], is more general than the original inequality that Bell derived. If one puts $a' = b' = c$, this inequality reduces to the original one that

$$|\xi(a,b) - \xi(a,c)| \leq 1 + \xi(c,b) \; . \tag{2}$$

According to Bell, the 'vital assumption is that the result b_β for particle 2 does not depend on the setting α, of the magnet(i.e. measuring apparatus) for particle 1, nor a_α on β,' which is usually taken as a requirement due to locality.

In quantum mechanics, however, there exist a state for which

$$\mathcal{B}_{\rm QM} > 2 \; . \tag{3}$$

Therefore we can conclude that 'for at least one quantum mechanical state, the statistical predictions of quantum mechanics is incompatible with separable predetermination' *à la* Bell.

With certain inspection regarding the basic presumptions in the above theorem, Bell[1] concluded that

> **(Interpretation of Bell theorem 1)**
>
> In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. ([2], p. 20)

One of the important concern of Bell was the issue of (im)possibility of the hidden variable theory, which can be defined to be the physical theory in which 'the quantum mechanical states can be regarded as ensemble of states further specified by additional variables, such that given values of these variables together with the state vector determine precisely the results of individual measurement.' With this definition, one can say

> **(Interpretation of Bell theorem 2)**
>
> The quantum-mechanical result cannot be reproduced by a hidden variable theory which is local. ([2], p. 38)

Shimony declares a stronger version:

> **(Interpretation of Bell theorem 3)**
>
> No physical theory satisfying the specified independence conditions can agree in all circumstances with the predictions of quantum mechanics. ([5], p. 90)

For the case that the independence conditions comes from the theory of relativity as locality requirement, one can state

(Interpretation of Bell theorem 4)

No local physical theory can agree in all circumstances with the predictions of quantum mechanics. ([5], p. 91)

All of these "received" views on Bell theorem emphasize that the assumption of locality makes main discrepancy from quantum mechanical prediction. Logically, however, Bell theorem has as its presupposition *both* the separability *and* the reality of physical quantities, as well as other subsidiary assumptions. As Einstein expressed his firm belief that 'the real factual situation of the system $\mathcal{S}_2$ is independent of what is done with the system $\mathcal{S}_1$, which is spatially separated from the former', it would be a better policy to examine the other presuppositions before proceeding to find fault with the assumption of locality. One of the main concern of present paper is to examine the validity of these interpretations of Bell theorem, on the basis of clearer analysis of "locality" and "reality".

3 Derivations of Bell's inequality from the reality of physical quantities

The significance of Bell theorem stems not only from its conflict with quantum mechanics, but also from its derivation under certain conditions, irrespectively of any particular dynamics. Therefore, once Bell's inequality could be derived under a clearly specified set of presumptions, at least one of the presumptions can be blamed to be "wrong". The main problem is how to find out the most responsible presumptions to be blamed.

The strategy we are taking in this paper is to separate the presumably possessed values of physical quantities from the actually measured values and see clearly how they are involved in the process of deriving Bell's inequality. By doing this, we find that the most responsible, hence vital, presumption underlying the derivation of Bell's inequality is the very assumption that the system possesses the values of physical quantities. We will therefore examine very carefully the possible consequence of the *assumption that there exist the possessed values of physical quantities for the system, irrespective of quantum mechanics.* The values A, B, A', B' are presumably possessed by the system, and the actual measured values of them a, b, a', b' may or may not be different from them.

Note that while our consideration resembles the standard discussion of Bell's inequality, the sole vital assumption here is that the system possesses the values of physical quantities.

3.1 Case 1

We first consider the case where the measured values a, b, are numerically equal to A, B, which are assumed to take either $+1$ or -1. The number of possible combinations for 4 outcomes a, a', b, b' is 2^4.

Now assume that the measured outcomes are the mere uncovering of the values of physical quantities which the system "possesses".

In this case, *whatever possible nonlocality is undergoing within the system*, the coincidence rate is given by

$$\xi(a,b) = \frac{1}{N}\sum_{i=1}^{16} a_i b_i N_i \ , \tag{4}$$

where a_i and b_i denote possible value of the measured outcomes a and b respectively.

From the positive-definiteness of probability, Bell's inequality is derived *à la* Wigner[6](See Appendix for detail.).

$$\mathcal{B} := |\xi(a,b) - \xi(a,b')| + |\xi(a',b) + \xi(a',b')| \leq 2 \tag{5}$$

It should be noticed that the only assumption made here is that the system eventually came to possess four values A, B, A', B' without any regard how they come to be existent and the measured outcomes are numerically equal to these possessed values. The possible correlations are fully allowed, and even the putative nonlocal influences between them are not excluded, since all we need is the outcomes which can be measured explicitly.

3.2 Case 2

We now consider the more general case where the actual measured values a, b of certain observables **A**, **B** can be unequal to the presumably possessed values A, B. Suppose that a and b can depend also on α, β, the parameters representing the measuring apparatus (such as direction of the magnet), and on whatever hidden parameters λ, and that the coincidence rate $\xi(a,b)$ for the measurement a, b is given by the expectation value of the product of two possessed values

$$\xi(a,b) = \sum_{\lambda} A(\alpha,\lambda) B(\beta,\lambda) P(A,B|\alpha,\beta,\lambda) \ , \tag{6}$$

where $P(A,B|\alpha,\beta,\lambda)$ is the joint probability for the event that the values $A(\alpha,\lambda)$ and $B(\beta,\lambda)$ to be observed as a and b, respectively, given the the parameters of apparatus and some hidden parameters. By this, we permit the possibility of $\xi(a,b)$ depending on the setting of the measuring apparatus and also unknown hidden parameters λ. Without loss of generality we can choose the values of four physical quantities as lying in $[-1,1]$, which includes the dichotomic case.

In order to derive Bell's inequality, it is necessary to assume the "factorizability" condition

$$P(A,B|\alpha,\beta,\lambda) = P_1(A|\alpha,\lambda) P_2(B|\beta,\lambda) p(\lambda) \ , \tag{7}$$

where $P_1(A|\alpha,\lambda)$ and $P_2(B|\beta,\lambda)$ are the probability distribution restricted to each subsystem and $p(\lambda)$ is the unknown, but arbitrary, distribution for hidden parameters. The condition Eq. (7) requires both the value independence between A and B, and the parameter independence between α and β. As it was noticed in Case 1, when we assume that the system possesses the four values A, B, A', B', all the possible correlations between them are fully taken account, and even the possible nonlocal influences, if any, might have been taken account in the process of possessing such values. It is therefore quite natural to require the value independence between A and B in this context. [Otherwise, the measurement of the value A would lose any meaning because quantities other than A (not only B but any value in the world) would be involved in the process.] Also the assumption of parameter independence between α and β is rather benign. If we have to allow the parameter dependence, no meaningful experiment can be performed because any instrument located at any place in the universe may influence the values in a undetermined way, even if we disregard the celebrated 'relativistic causality'.

From the assumption of factorizability, Eq. (7), the coincidence rate is given by

$$\xi(a,b) = \sum_{\lambda} A(\alpha,\lambda) P_1(A|\alpha,\lambda) B(\beta,\lambda) P_2(B|\beta,\lambda) p(\lambda) = \sum_{\lambda} \tilde{A}_{\alpha}(\lambda) \tilde{B}_{\beta}(\lambda) p(\lambda) \tag{8}$$

where $\tilde{A}_\alpha(\lambda) = A(\alpha,\lambda)P_1(A|\alpha,\lambda)$ and $\tilde{B}_\beta(\lambda) = B(\beta,\lambda)P_2(B|\beta,\lambda)$, which do not exceed unity. For these four numbers, it holds that(See Appendix for detail.)

$$\mathcal{Q} := |\tilde{A}_\alpha \tilde{B}_\beta - \tilde{A}_\alpha \tilde{B}_{\beta'}| + |\tilde{A}_{\alpha'} \tilde{B}_\beta + \tilde{A}_{\alpha'} \tilde{B}_{\beta'}| \leq 2\ . \tag{9}$$

From this inequality, we obtain

$$\mathcal{B} := |\xi(a,b) - \xi(a,b')| + |\xi(a',b) + \xi(a',b')| = \sum_\lambda \mathcal{Q}p(\lambda) \leq \mathcal{Q}_{\max} \sum_\lambda p(\lambda) \leq 2\ . \tag{10}$$

where $\mathcal{Q}_{\max}$ means the maximum value of $\mathcal{Q}$.

Note that Eq. (7) can be compared with Eq. (10) of Ref. [7]. In the latter the probability distribution $p(\lambda)$ for the hidden parameters is absent. Though one can absorb $p(\lambda)$ mathematically into the definition of $\tilde{A}$ or $\tilde{B}$, its physical meaning is not unimportant. While we do not allow $P_1(A|\alpha,\lambda)$ to be dependent on B or β, we don't have to care about the way how two subsystems happen to possess the values A and B respectively. This ignorance is generally expressed by a probability distribution for the unknown parameters. In this sense $p(\lambda)$ shows that a sort of nonlocality is allowed, so long as it is *not* observed at all and is prior to any measuring process.

3.3 A variation of Case 2

We examine here a variation of the previous case. Instead of the factorizability condition, Eq. (7), we can assume that the joint probability distributions for two physical quantities are not dependent on specific parameter adjust for measuring apparatus, viz.,

$$P(A,B|\alpha,\beta,\lambda) = P(A,B|\alpha,\beta',\lambda) = P(A,B|\alpha',\beta,\lambda) = P(A,B|\alpha',\beta',\lambda)\ . \tag{11}$$

Invoking this assumption at the first step of Eq. (A.21)(See Appendix for detail.), we arrive at Bell's inequality. This is a rather strong assumption, since it means that the joint probability of of the distant events does not depend on the settings of the measuring apparatus at all. In fact, it is stronger than our factorizability requirement, Eq. (7). In his own derivation of the inequality, Bell employed the above assumption, Eq. (11) implicitly, in addition to the explicit locality requirement, Eq. (7).

4 Alternative interpretation of Bell theorem

In the previous section, we considered the consequences of the assumption that the system "possesses" the values of physical quantities. For the Case 1, in which the measured values are assumed to be equal to the values which the system possesses, one can derive Bell's inequality, fully permitting any sort of nonlocality. But, for the Case 2, in which the measured values are not equal to the values presumably possessed by the system, some added assumption is need to arrive at the Bell's inequality. In this case, one should invoke either the 'factorizability' condition, Eq. (7), or the strong assumption of the identical probability for the joint event of each pair, Eq. (11). But are these additional assumptions really matter? What kind of locality does it demand or prohibit? Are the interpretations mentioned in Sec. 2 still valid?

In order to discuss these questions, it might perhaps be convenient to separate the concept of nonlocality into two parts: one the ontological one, and the other, the epistemological one. By the *ontological nonlocality*

we mean any possible nonlocal influences engendered in the process of determining the presumably existent values A, B, A', B' in the system, and by the *epistemological nonlocality*, the presumable nonlocal influences occurring in the measuring process of obtaining the values a, b, a', b' from the existent values A, B, A', B'.

The factorizability condition, Eq. (7), for instance, implies that we would have to allow the epistemological nonlocality if we uphold the fact that there exist the values of physical quantities, which are different from the measured values of them but somehow possessed by the system.

Using the above terminology, we summarize our conclusion as following: the measured values a, b, a', b' corresponding to physical observables $\mathbf{A}$, $\mathbf{B}$ cannot be claimed to be existent in the system in general no matter what nonlocality is assumed, and the possessed values A, B, A', B' corresponding to $\mathbf{A}$, $\mathbf{B}$ which are different from but somehow related to the measured values a, b, a', b' cannot be claimed to be existent in the system unless the highly unprobable epistemological nonlocality is assumed. This shows that it is not legitimate for the system to possess the set of observed values. Moreover, it is also illegitimate to have the set of presumed values related to the actual measured ones via possible parameters of the measuring apparatus or other hidden variables, unless we assume such a weird situation that any physical values or instrumental setups space-likely separated from the measuring activity can influence the measurement result.

On the other hand, the abandonment of the conception of the possessed physical values does not cost at all in physical theories. The concept of possession of observable values by the system, or the concept of the existence of such values within the system does not have any significance even in classical physics. What is needed instead is the concept of the *state* of the system. In classical mechanics, for instance, the *state* of the system is represented by the pair of measured values, *not possessed values*, of position and momentum, which is used *only* as the initial conditions for the prediction of future events. It utterly does not concern whether such values are possessed by the system or not. In other words, the conception of possessed physical values is fully redundant in classical mechanics.

In quantum mechanics, however, the conception of possessed physical values is no longer redundant but contradictory to the concept of the state. As we already have seen, we have to accept an absurd conception like epistemological nonlocality just to save this conception without gaining any benefit in the description of physical world.

Recently Ref.[8] discussed a refutation of Bell theorem, maintaining that one cannot compare an a local realistic inequality based on Eq. (11), which implies that 'all four pairs of directions are considered simultaneously relevant to each particle pair', with the quantum mechanical prediction derived from "weakly objective interpretation" which permits different joint probability distributions for four pairs of observables. For 'there is no way to derive a Bell inequality' within the weakly objective interpretation. However, one can derive Bell's inequality with a weaker presumption of the factorizability, Eq. (7) without assuming Eq. (11). This shows very well that 'within a strictly quantum mechanical framework' only the *state* of the system is considered and there is no place for the assumption of "possession of the values of physical quantities". We see that the conflict is *not* between local hidden variable theory and quantum mechanics, *but* between the assumption of reality of physical quantities and the experiment.

Our consideration in this paper is deeply related to the general investigations on the foundations of dynamics in totality and can be extended to an alternative interpretation of quantum mechanics, as well as other dynamics, within a more systematic framework of meta-dynamics[11, 12, 13, 14].

Appendix: Detailed derivations of Bell inequality

A.1 Quantum-mechanical result

As a typical example of the composite system of two-state systems which violates Bell's inequality, consider two spin 1/2 particles in a singlet state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2\Big)\ . \tag{A.1}$$

For this state, the coincidence rate is given by

$$\xi_{\rm QM}(a,b) = -\cos(\alpha-\beta) \tag{A.2}$$

and

$$\mathcal{B}_{\rm QM} = |\cos(\alpha-\beta) - \cos(\alpha-\beta')| + |\cos(\alpha'-\beta) + \cos(\alpha'-\beta')|\ . \tag{A.3}$$

If we choose

$$|\alpha-\beta'| = 3\pi/4, \quad |\alpha-\beta| = |\alpha'-\beta| = |\alpha'-\beta'| = \pi/4\ , \tag{A.4}$$

the violation of Bell's inequality is maximal:

$$\mathcal{B}_{\rm QM} = 2\sqrt{2} > 2\ .$$

In the experiment[16] by A. Aspect, P. Grangier and G. Roger, for instance, the quantum-mechanical prediction is

$$\mathcal{B}_{\rm QM} = 2.70 \pm 0.05\ , \tag{A.5}$$

with the incompleteness in the apparatus considered, whereas the experimental result is

$$\mathcal{B}_{\rm exp} = 2.6970 \pm 0.015 \tag{A.6}$$

in great agreement with the quantum-mechanical prediction.

However, the above calculation has a loophole with the compatibility of the associated observables. Consider explicitly four self-adjoint operators acting on $\mathfrak{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, such as

$$A = \vec{\sigma}\cdot\mathbf{n}_\alpha \otimes 1\!\!1\ , \quad A' = \vec{\sigma}\cdot\mathbf{n}_{\alpha'} \otimes 1\!\!1\ , \quad B = 1\!\!1 \otimes \vec{\sigma}\cdot\mathbf{n}_\beta\ , \quad B' = 1\!\!1 \otimes \vec{\sigma}\cdot\mathbf{n}_{\beta'}\ . \tag{A.7}$$

Note that while $[A,B] = [A,B'] = [A',B] = [A',B'] = 0$, it is no longer the case for the pairs (A,A') and (B,B'). The above derivation demands that the system should *possess* the values (A,A',B,B') simultaneously, which is forbidden for quantum mechanics.

A.2 Detailed proof for Case 1

Consider four physical quantities A_α, $A_{\alpha'}$, B_β, $B_{\beta'}$, which are assumed to be two-valued, $+1$ or -1. Denote the outcomes of these physical quantities by a, a', b, b', respectively. If these outcomes are just the values which the system "possessed" prior to measurement, the coincidence rate for pairs of physical quantities are plainly given by the simple expectation value of the products of these.

The number of the possible combination of the products is 16. Let the frequency for each case N_i ($i = 1, \cdots, 16$) and the corresponding probability p_i. Then for N measurements,

$$p_i \equiv \frac{N_i}{N}\ , \quad N = N_1 + N_2 + \cdots + N_{16}\ , \quad 0 \leq p_i \leq 1\ , \quad i = 1, \cdots, 16 \tag{A.8}$$

It is convenient to have the following table:

	a	b	a'	b'	ab	ab'	$a'b$	$a'b'$
1	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
2	$+$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
3	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$
4	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$
5	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$+$
6	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$-$
7	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
8	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
9	$-$	$+$	$+$	$+$	$-$	$-$	$+$	$+$
10	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$
11	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$-$
12	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$+$
13	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$+$
14	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$-$
15	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$-$
16	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$

Here the value of the column of ab is just the product of two values of a and b. Hence the value of $\xi(a,b)$, for instance, is given by

$$\begin{aligned}\xi(a,b) = p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 \\ - p_9 - p_{10} - p_{11} - p_{12} + p_{13} + p_{14} + p_{15} + p_{16}\end{aligned} \tag{A.9}$$

and similarly for $\xi(a,b')$, $\xi(a',b)$ and $\xi(a',b')$.

Since the difference of two arbitrary positive number cannot be larger than their sum, we have

$$\begin{aligned}|\xi(a,b) - \xi(a,b')| &= 2(p_2 + p_4 - p_5 - p_7 - p_{10} - p_{12} + p_{13} + p_{15}) \\ &\leq 2(p_2 + p_4 + p_5 + p_7 + p_{10} + p_{12} + p_{13} + p_{15})\end{aligned} \tag{A.10}$$

and similarly

$$\begin{aligned}|\xi(a',b) + \xi(a',b')| &= 2(p_1 - p_3 - p_6 + p_8 + p_9 - p_{11} - p_{14} + p_{16}) \\ &\leq 2(p_1 + p_3 + p_6 + p_8 + p_9 + p_{11} + p_{14} + p_{16})\ .\end{aligned} \tag{A.11}$$

Therefore we obtain

$$\mathcal{B} := |\xi(a,b) - \xi(a,b')| + |\xi(a',b) + \xi(a',b')| \leq 2 \sum_{i=1}^{16} p_i = 2\ . \tag{A.12}$$

A.3 Detailed proof of Case 2

For arbitrary two real numbers y, y' such that $|y| \leq 1$, $|y'| \leq 1$,

$$|y \pm y'| \leq 1 \pm yy' \ , \tag{A.13}$$

because

$$|y \pm y'|^2 - (1 \pm yy')^2 = y^2(1 - y'^2) + y'^2 - 1 \leq 1 - y'^2 + y'^2 - 1 = 0 \ , \tag{A.14}$$

where the last step results from $|y| \leq 1$.

Using Eq. (A.13), for four real numbers x, x', y, y' whose absolute values are not larger than 1,

$$|xy - xy'| = |x| \cdot |y - y'| \leq |x|(1 - yy') \leq 1 - yy' \ , \tag{A.15}$$

$$|x'y + x'y'| = |x'| \cdot |y + y'| \leq |x'|(1 + yy') \leq 1 + yy' \tag{A.16}$$

and hence

$$|xy - xy'| + |x'y + x'y'| \leq (1 - yy') + (1 + yy') = 2 \ . \tag{A.17}$$

For four numbers $\tilde{A}_\alpha$, $\tilde{A}_{\alpha'}$, $\tilde{B}_\beta$, $\tilde{B}_{\beta'}$ the magnitude of which do not exceed unity, one has

$$\mathcal{Q} := |\tilde{A}_\alpha \tilde{B}_\beta - \tilde{A}_\alpha \tilde{B}_{\beta'}| + |\tilde{A}_{\alpha'} \tilde{B}_\beta + \tilde{A}_{\alpha'} \tilde{B}_{\beta'}| \leq 2 \ . \tag{A.18}$$

Using the “factorizability” condition, Eq. (7), Bell’s inequality is derived.

A.4 Detailed proof of a variation of Case 2

We give another derivation of Bell’s inequality, motivated by a recent proof[17]. We indicated the intermediate steps in order to make the assumptions more explicit. As in Sec. A.3, we have

$$|A_\alpha B_\beta - A_\alpha B_{\beta'}| + |A_{\alpha'} B_\beta + A_{\alpha'} B_{\beta'}| \leq 2 \ . \tag{A.19}$$

for four random variables A_α, $A_{\alpha'}$, B_β, $B_{\beta'}$ whose ranges lie in $[-1, 1]$.

Since for $X \in [-1, 1]$,

$$|E(X)| = \Big| \sum_i X_i p_i \Big| \leq \sum_i |X_i| p_i = E(|X|) \ , \tag{A.20}$$

we have

$$\begin{aligned}
&|E(A_\alpha B_\beta) - E(A_\alpha B_{\beta'})| + |E(A_{\alpha'} B_\beta) + E(A_{\alpha'} B_{\beta'})| \\
&\qquad = |E(A_\alpha B_\beta - A_\alpha B_{\beta'})| + |E(A_{\alpha'} B_\beta + A_{\alpha'} B_{\beta'})| \\
&\qquad \leq E(|A_\alpha B_\beta - A_\alpha B_{\beta'}|) + E(|A_{\alpha'} B_\beta + A_{\alpha'} B_{\beta'}|) \\
&\qquad = E(|A_\alpha B_\beta - A_\alpha B_{\beta'}| + |A_{\alpha'} B_\beta + A_{\alpha'} B_{\beta'}|) \\
&\qquad \leq \max(|A_\alpha B_\beta - A_\alpha B_{\beta'}| + |A_{\alpha'} B_\beta + A_{\alpha'} B_{\beta'}|) \\
&\qquad \leq 2 \ .
\end{aligned} \tag{A.21}$$

The equalities in the first and third line comes from the linearity of expectation value. The inequality in the second line is Eq. (A.20) and the fourth line results from

$$E(Y) \leq \sum_i Y_{\max} p_i = Y_{\max} \sum_i p_i = Y_{\max} \tag{A.22}$$

where $Y_{\max}$ is the maximum value which the random variable Y can assume. We used the fact $0 \leq p_i \leq 1$. The last line of Eq. (A.21) is Eq. (A.19).

The coincidence rate of two physical quantities is given by the expectation value of the joint event,

$$\xi(a,b) = E(A_\alpha B_\beta) \; . \tag{A.23}$$

Substituting this into Eq. (A.21), we get

$$\mathcal{B} := |\xi(a,b) - \xi(a,b')| + |\xi(a',b) + \xi(a',b')| \leq 2 \; . \tag{A.24}$$

Note that the crucial assumption in the first and third line of Eq. (A.21) is that

$$P(A_\alpha, B_\beta) = P(A_\alpha, B_{\beta'}) = P(A_{\alpha'}, B_\beta) = P(A_{\alpha'}, B_{\beta'}) \; , \tag{A.25}$$

for all four parameters α, β, α', β'.

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